Towards Breaking the Exponential Barrier for General Secret Sharing

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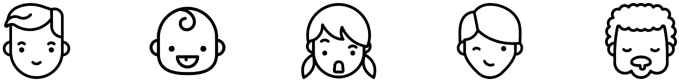
CNRS and ENS

May 6, 2018

Secret Sharing [Blakley'79,Shamir'79,Ito-Saito-Nishizeki'87]

Secret $s \in \{0, 1\}$



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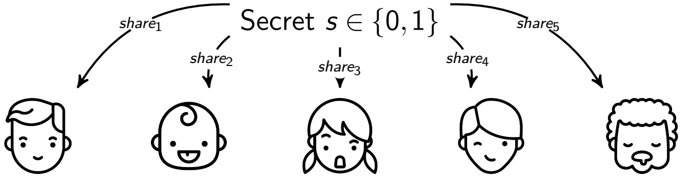

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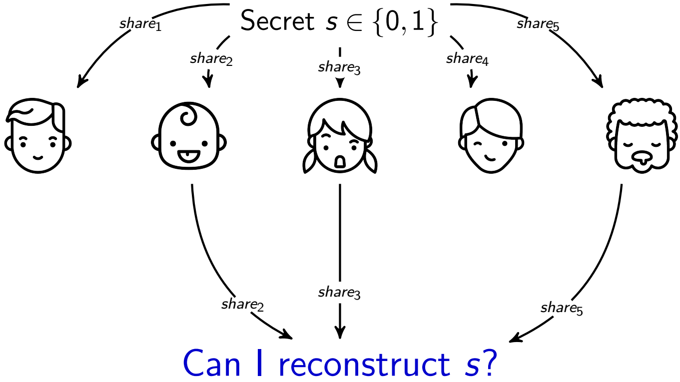

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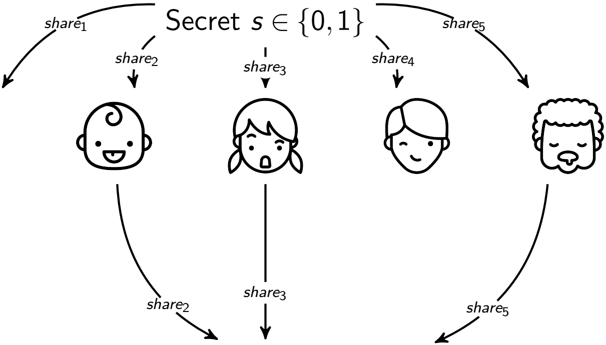

Can I reconstruct s?

Threshold Secret Sharing [Shamir'79]

YES if I gets $\geq t$ shares;

NO INFO if I gets $< t$ shares.

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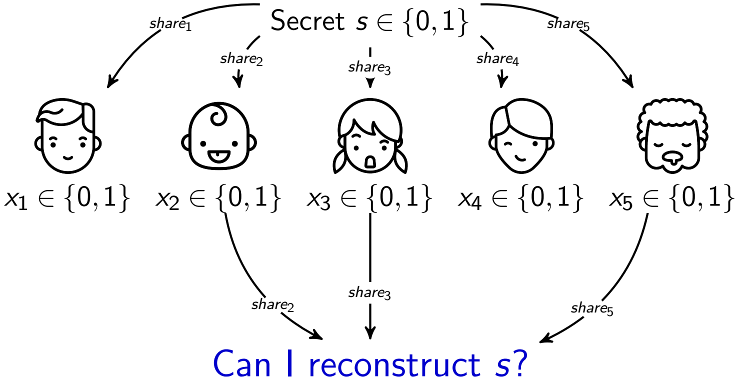

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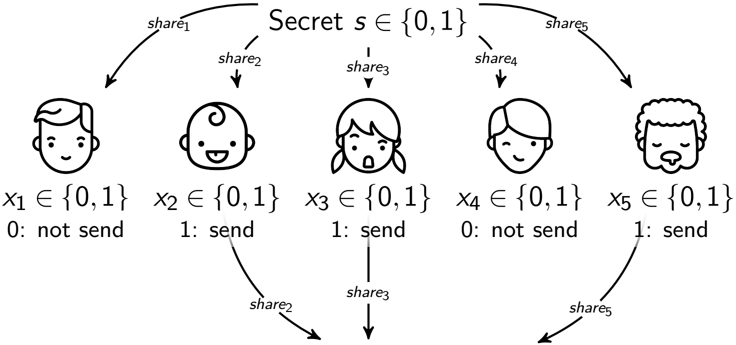

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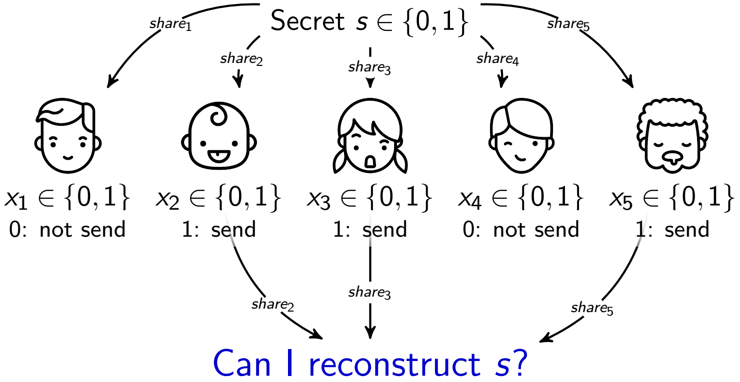

Threshold Secret Sharing [Shamir'79]

YES if $\text{threshold}_t(x_1, \ldots, x_n) = 1$;

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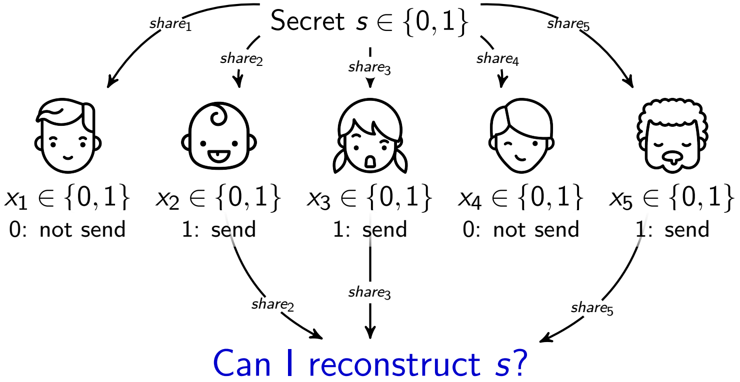

General Secret Sharing [ISN'89] monotone $F : \{0,1\}^n \to \{0,1\}$

YES if $F(x_1, \ldots, x_n) = 1$;

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Key Complexity Measure: Total Share Size

Best Known Secret Sharing Schemes

Share size $\leq O(\text{monotone formula size}) \leq \tilde{O}(2^n)$. [Benaloh-Leichter'88]
Share size $\leq O(\text{monotone span program size}) \leq \tilde{O}(2^n)$. [Karchmer-Wigderson'93]

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Lower Bounds

$\exists F$ that share size $\geq \tilde{O}(2^{n/2})$ for *linear* secret sharing. [KW'93]
$\exists F$ that total share size $\geq \tilde{\Omega}(n^2)$. [Csirmaz'97]

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Empirical Observation: In general secret sharing, share size grows (polynomially) on representation size.

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Representation Size Barrier?

For any collection of $2^{2^{\Omega(n)}}$ monotone access functions,
$\exists F$ in the collection that requires $2^{\Omega(n)}$ share size.

Our results

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Our Theorem: Overcoming the Representation Size Barrier

There is a collection of $2^{2^{n/2}}$ monotone access functions, s.t.
$\forall F$ in the family has a secret sharing scheme with share size $2^{\tilde{O}(\sqrt{n})}$.

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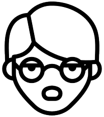
Main Tool: Multi-party Conditional Disclosure of Secrets (CDS)

Multi-party CDS scheme with communication complexity $2^{\tilde{O}(\sqrt{n})}$.

Multi-party Conditional Disclosure of Secrets

[Gertner-Ishai-Kushilevitz-Malkin'00]

$x_1 \in \{0,1\}$ $x_2 \in \{0,1\}$ $x_3 \in \{0,1\}$ $\cdots$ $x_n \in \{0,1\}$



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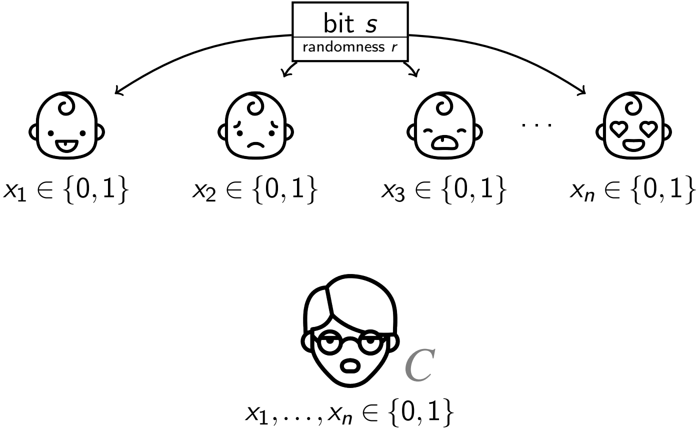

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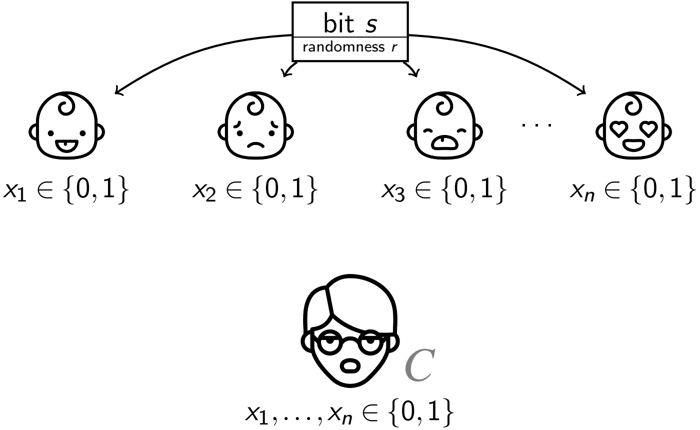

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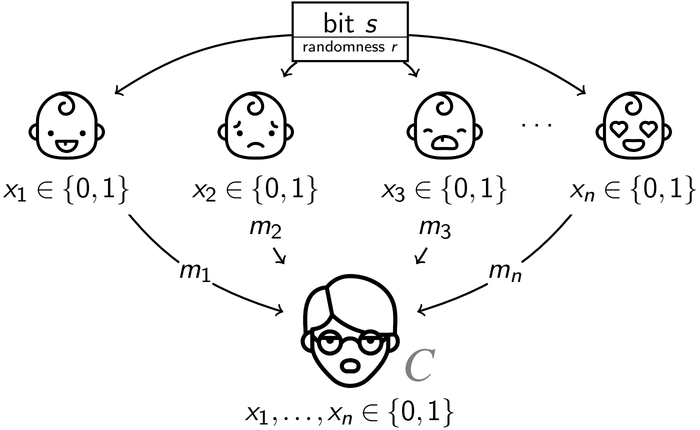

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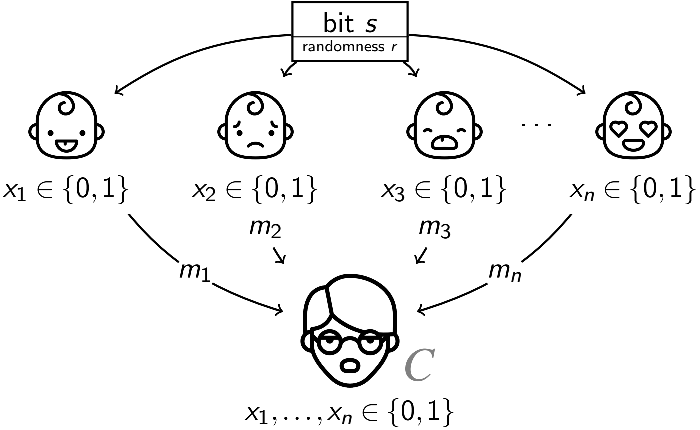

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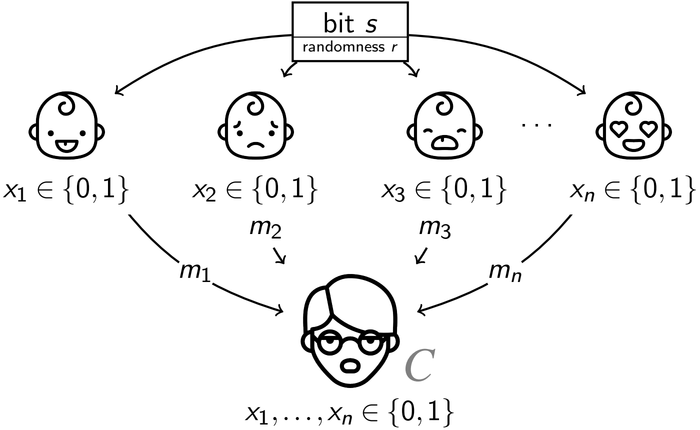

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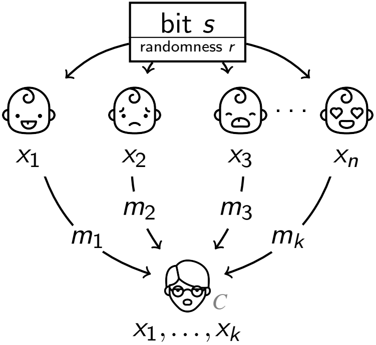
Multi-party Conditional Disclosure of Secrets [GIKM'00]

Multi-party CDS



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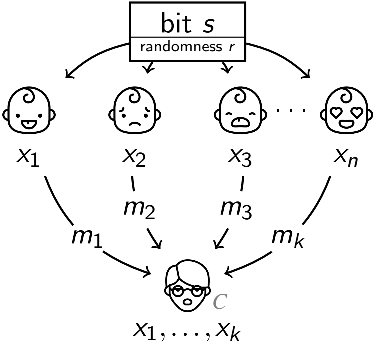

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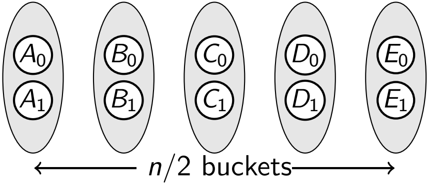

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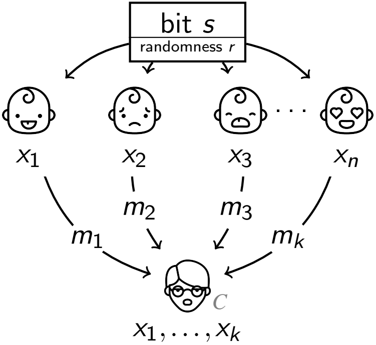

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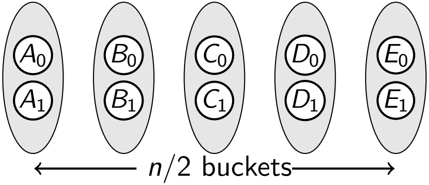

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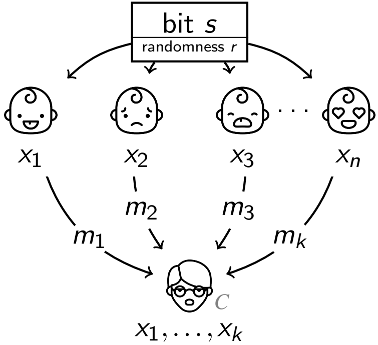

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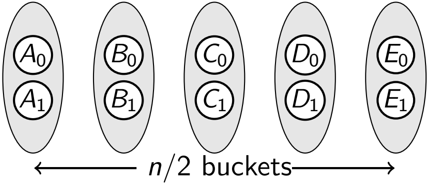

- Promise: Exactly one participant from each bucket
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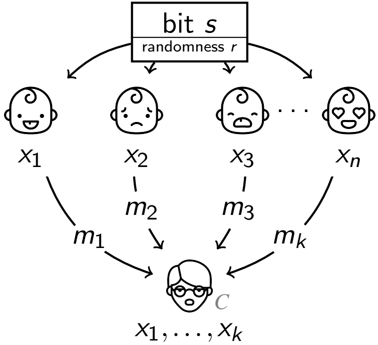

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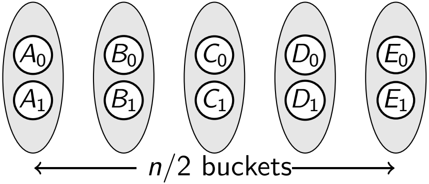

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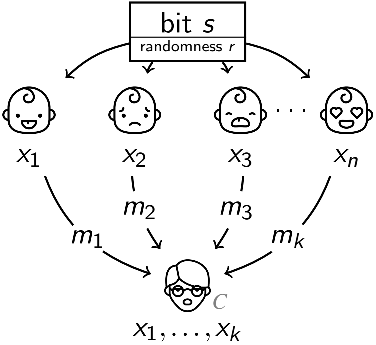

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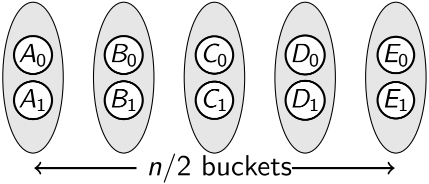

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- A_0's share $= m_1(0, s, r)$,
  A_1's share $= m_1(1, s, r)$, etc

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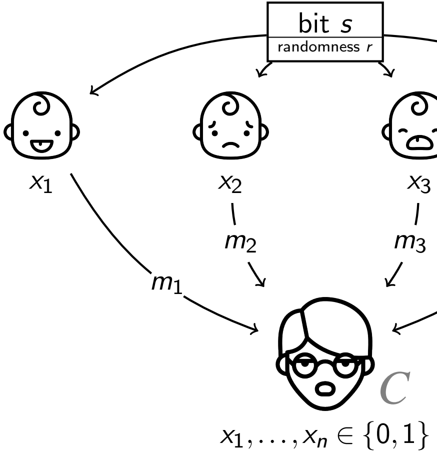

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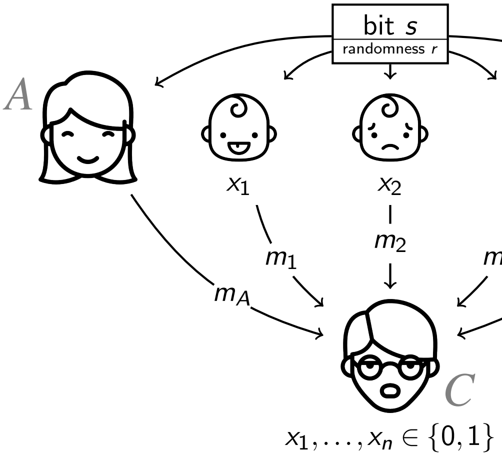

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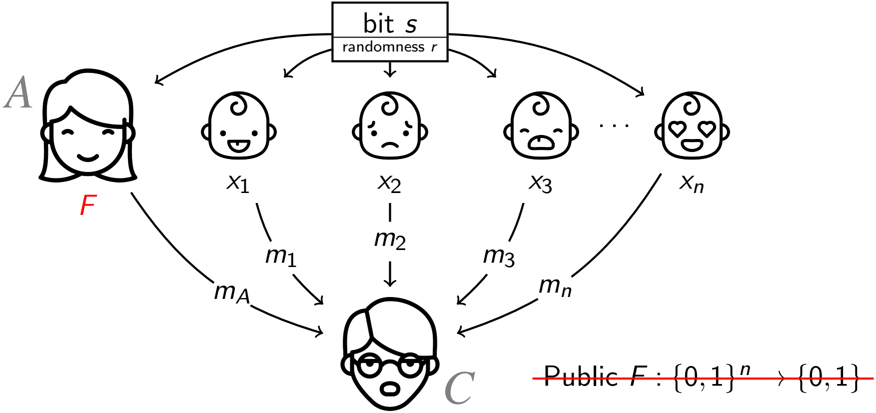

$F \in \{0,1\}^{2^n}$, $x_1, \dots, x_n \in \{0,1\}$

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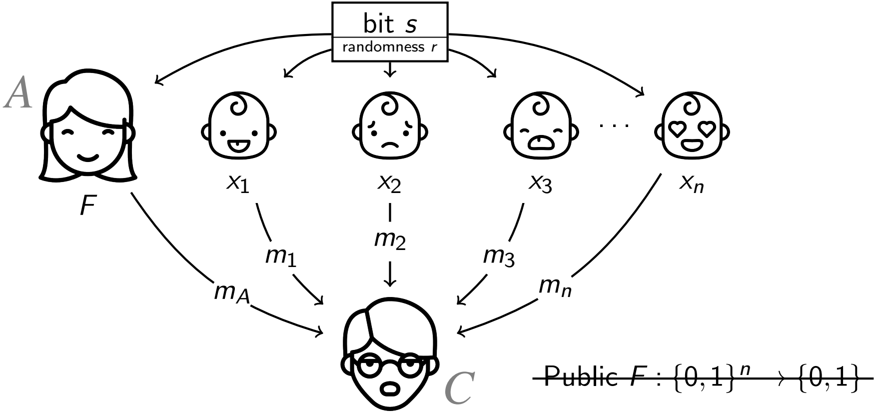

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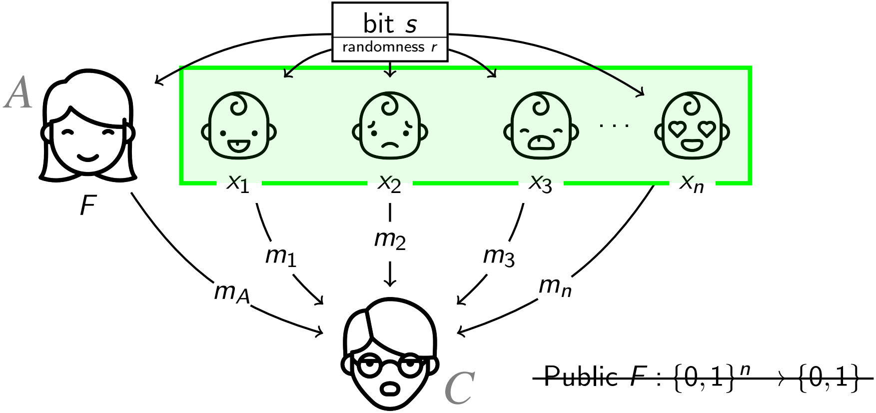

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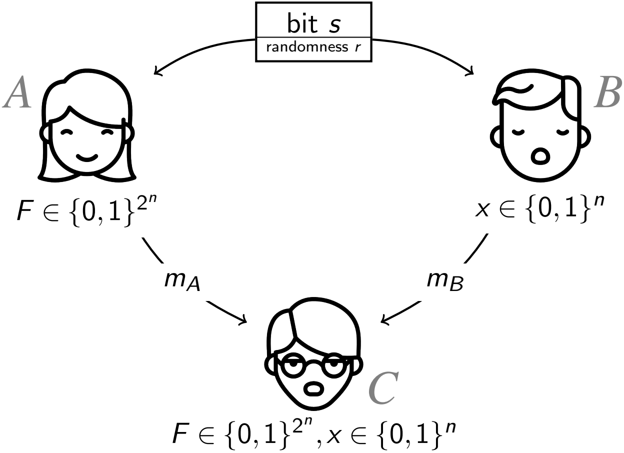
2-party Conditional Disclosure of Secrets [GIKM'00]



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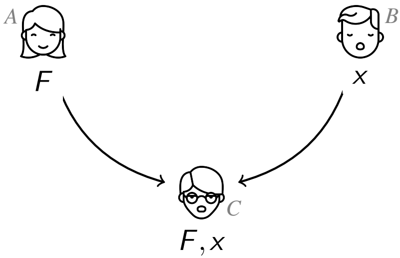
2-party CDS: Previous Works

2-Party CDS

Communication Complexity		Reconstruction
$\Theta(2^{n/2})$	[GKW'15]	linear
$\Theta(2^{n/3})$	[LVW'17]	quadratic
$2^{\tilde{O}(\sqrt{n})}$	[LVW'17]	general
$\Omega(n)$	[GKW'15]	general

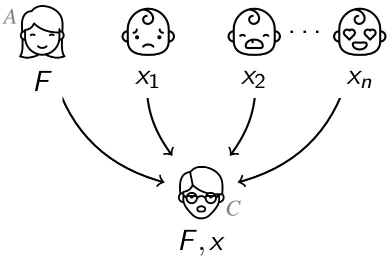
2-party CDS $\implies$ Multi-party CDS

2-party CDS



- $O(2^{n/2})$ linear reconstruction [GKW'15]
- $O(2^{n/3})$ quadratic reconstruction [LVW'17]
- $2^{\tilde{O}(\sqrt{n})}$ general reconstruction [LVW'17]

Multi-party CDS



2-party CDS $\Longrightarrow$ Multi-party CDS

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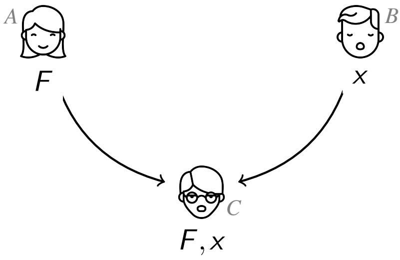

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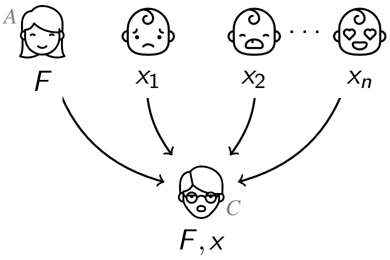

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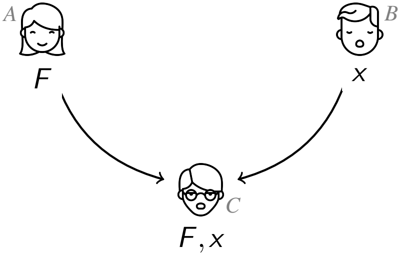
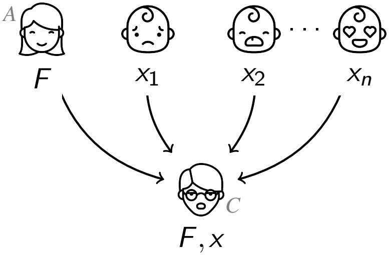


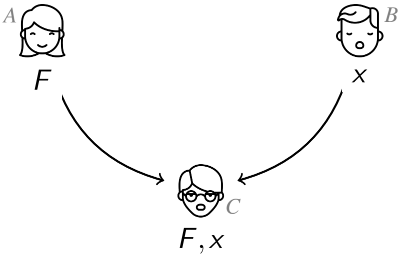
Multi-party CDS



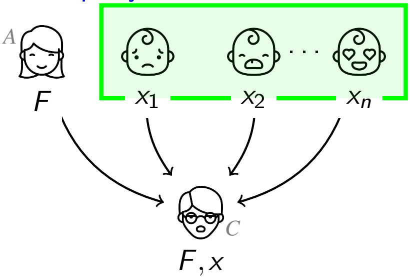
- $O(2^{n/2})$ linear reconstruction [GKW'15] $\longrightarrow$ $O(2^{n/2})$ linear reconstruction
- $O(2^{n/3})$ quadratic reconstruction [LVW'17] $\longrightarrow$ $O(2^{n/3})$ quadratic reconstruction
- $2^{\tilde{O}(\sqrt{n})}$ general reconstruction [LVW'17] $\longrightarrow$ $2^{\tilde{O}(\sqrt{n})}$ general reconstruction

2-party CDS $\Longrightarrow$ Multi-party CDS

2-party CDS



Multi-party CDS



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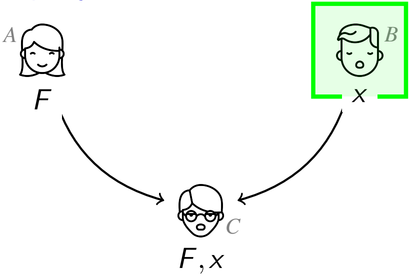

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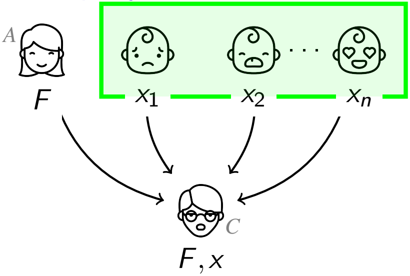

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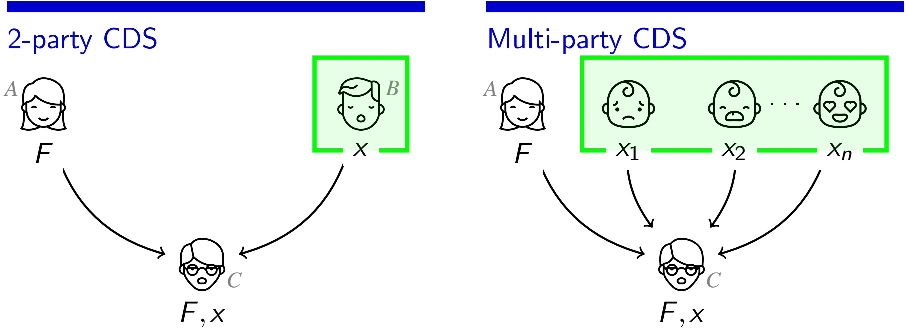
- $2^{\tilde{O}(\sqrt{n})}$ general reconstruction [LVW'17] $\longrightarrow$ $2^{\tilde{O}(\sqrt{n})}$ general reconstruction

2-party CDS $\Longrightarrow$ Multi-party CDS



Key Idea: Player Emulation [Hirt-Maurer'00]

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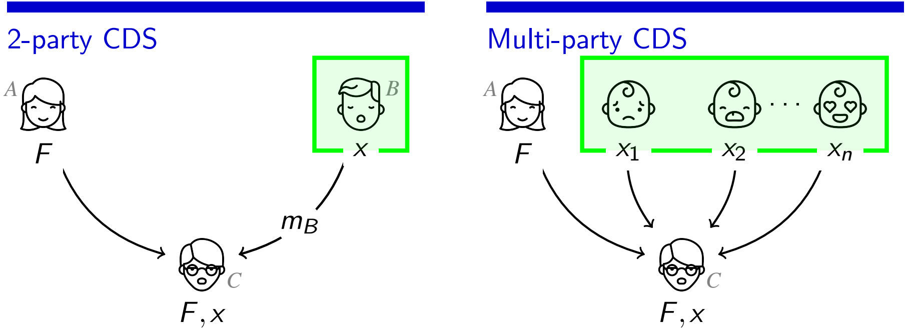

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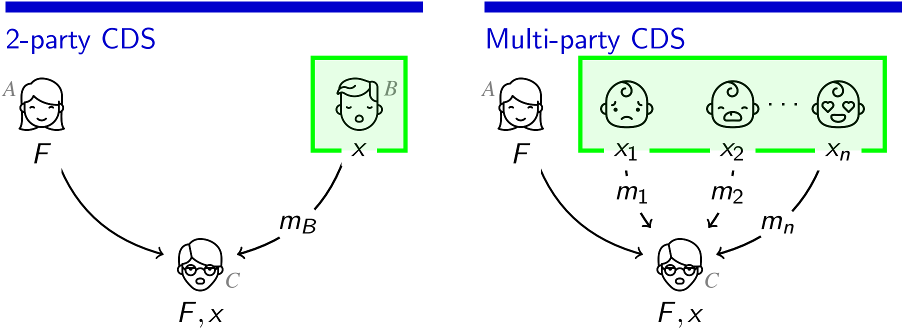

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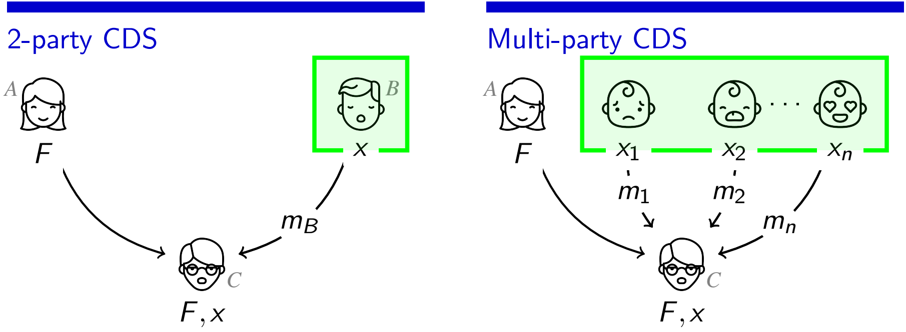

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- PSM (Private Simultaneous Messages) [FKN'94] $\approx$ Non-Interactive MPC

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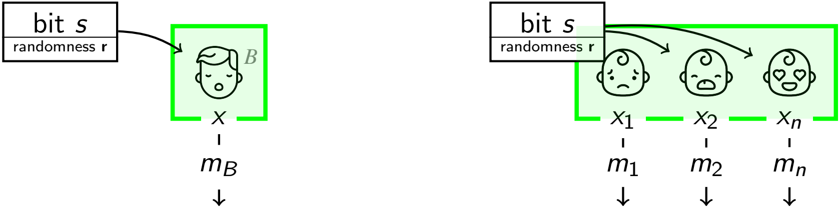

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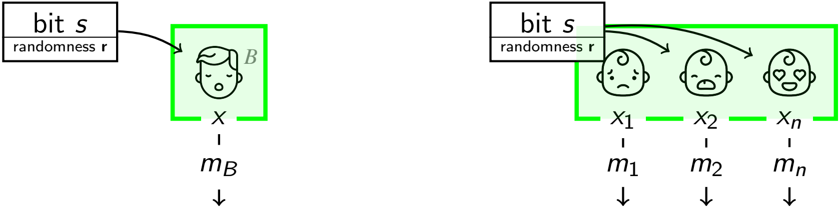

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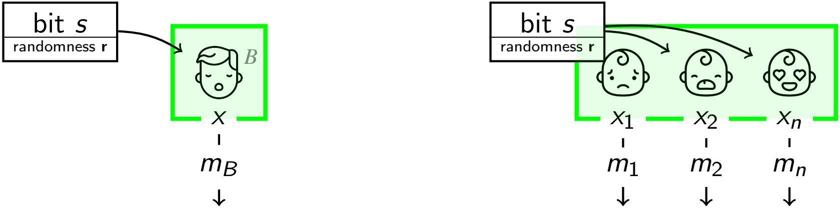

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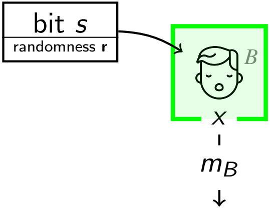

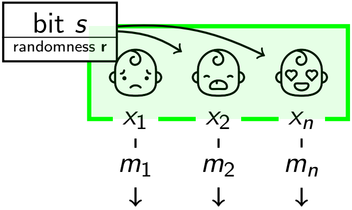

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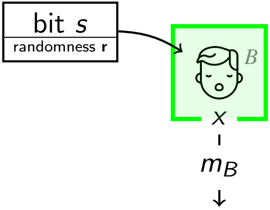

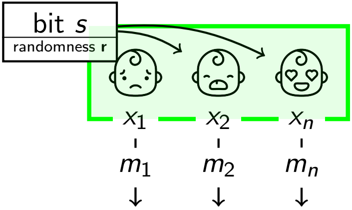

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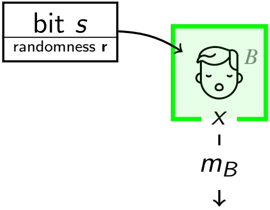

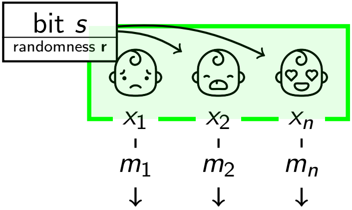

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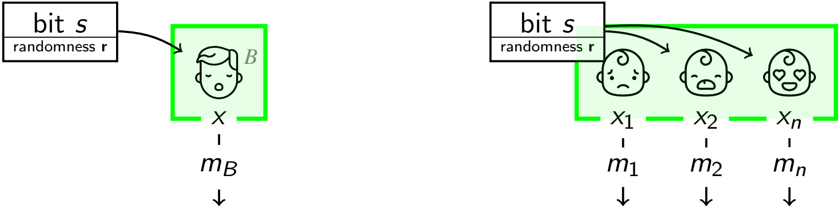
- If $m_B(x, s, \mathbf{r})$ computable by small arithmetic formula, PSM communication is small. [IK'02,AIK'04]
- Is $x \mapsto \mathbf{u}_x$ simple?

2-party CDS $\implies$ Multi-party CDS



New Construction of Matching Vectors

- mapping $x \mapsto \mathbf{u}_x$ computable by small formula

2-party CDS $\Longrightarrow$ Multi-party CDS

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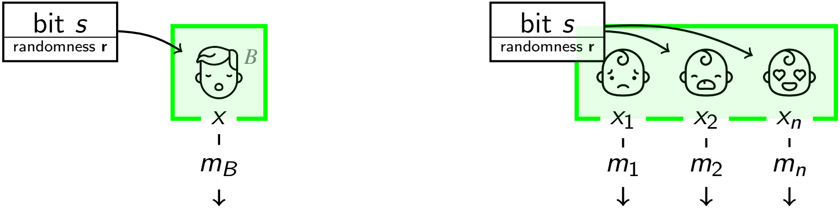

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2-party CDS $\implies$ Multi-party CDS

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2-party CDS $\Longrightarrow$ Multi-party CDS

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There is a multi-party CDS scheme with communication complexity $2^{O(\sqrt{n}\log n)}$ as long as the total input length is n bits.

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Secret sharing for double-exponentially many access functions

There is a collection of $2^{2^{n/2}}$ access functions, s.t.
$\forall F$ in the family has a secret sharing scheme with share size $2^{O(\sqrt{n}\log n)}$.

Our Results

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$O(2^{n/2})$ [GKW'15]

linear reconstruction

$O(2^{n/3})$ [LVW'17]

quadratic reconstruction

$2^{O(\sqrt{n}\log n)}$ [LVW'17] ⟶

general reconstruction

Multi-party CDS

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Our Results

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$O(2^{n/2})$ [GKW'15] → $O(2^{n/2})$ [This,BP'18]

linear reconstruction → linear reconstruction, optimal

$O(2^{n/3})$ [LVW'17] → $O(2^{n/3})$

quadratic reconstruction → quadratic reconstruction, optimal

$2^{O(\sqrt{n}\log n)}$ [LVW'17] → $2^{O(\sqrt{n}\log n)}$ [This]

general reconstruction → general reconstruction

Multi-party CDS

Subsequent Works on Secret Sharing

Secret sharing for even more access functions [This,BKN'18]

There is a collection of $2^{\binom{n}{n/2}}$ access functions, s.t.
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Subsequent Works on Secret Sharing

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$\updownarrow$

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Secret sharing for all access functions [LV'18 @STOC]

$\forall F$ has a secret sharing scheme with share size $2^{0.994n}$.

To Summarize

(or representation)

Can communication $\ll$ computation size?

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Computational

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(or representation)

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- What's next?